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Analysis of Three-Dimensional Optimal Evasion with Linearized Kinematics

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Three-dimensional optimal missile avoidance is analyzed with a linearized kinematic model. The solution requires maximum load factor and the problem is reduced to optimal roll position control having two phases: 1) orientation of the lift vector into the optimal evasion plane, 2) rapid 180 deg roll maneuvers governed by a switch function. For circular missile vectograms, the plane of optimal evasion is perpendicular to the line of sight. Evading from roll-stabilized missiles of rectangular vectogram, further advantage can be taken maximizing the target-missile maneuver ratio. Bounded roll-rate reduces the miss distance but does not affect the optimal evasive maneuver structure.

Nomenclature

a	= lateral acceleration
A	= system matrix, Eq. (14)
A_I	= single channel matrix, Eq. (15)
B	= control matrix, Eq. (18)
H	= variational Hamiltonian
J	= pay-off function, Eq. (48)
K_N	= constant of true proportional navigation, Eq. (9)
m	= miss distance, Eq. (46)
N'	= effective proportional navigation ratio, Eq. (9)
P_T	= roll rate (control variable)
R	= relative distance
S_1,S_2	= switch functions, Eqs. (55) and (68)
t	= time
t_f	= nominal time of flight, Eq. (8)
u	= control vector
V	= velocity
V_R	= closing speed, Eq. (6)
x_i	= state-vector components
y,z	= components of R, perpendicular to the initial line of sight, Eq. (12)
$\tilde{\gamma}_T$	= dynamic similarity parameter, Eq. (92)
δ_1,δ_2	= costate dependent coefficients, Eqs. (52) and (53)
θ	= normalized time-to-go, Eq. (63)
λ_i	= costate vector components
μ	= missile-target maneuver ratio, Eq. (30)
μ_I	= missile-target maneuver ratio, of a single channel, Eq. (33)
τ	= missile time constant
ϕ	= roll angle
$\tilde{\phi}_T$	= normalized target roll-rate limit
χ	= azimuth angle
Ω	= angular velocity

Superscripts

$(\vec{\ })$	= three-dimensional vector
$(\tilde{\ })$	= nondimensional variables
$(\ )^T$	= transpose of a matrix
$(\ )^*$	= optimal control functions
$(\dot{\ })$	= time derivative

Subscripts

c	= commanded value
f	= final value
i	= index
M	= missile
r	= required value
R	= line of sight
s	= saturation
T	= target
0	= initial value

Bold face type has been used for column vectors

Introduction

THE missile-aircraft pursuit-evasion problem can be either formulated as a zero-sum differential game, or decomposed to two reciprocal optimal-control problems whose respective objectives are to determine: 1) optimal guidance laws against maneuvering targets, and 2) optimal evasive maneuvers from guided missiles.

Regardless of the formulation, the problem is inherently complex. The relative pursuer-evader kinematics are expressed by a nonlinear three-dimensional vector equation. Both vehicles' dynamics are expressed by sets of nonlinear differential equations. Moreover, the guidance law of the pursuer is implemented by a rather complicated transfer function. The exact solution for each of the alternative formulations requires the solution of a nonlinear two-point boundary-value problem of very high dimension. The computation of such a solution, although feasible, is so time-consuming that it makes this approach impractical for systematic studies.

For a systematic analysis, which is necessary to create an insight into this complex problem, simplified analytic solutions are required. Important simplification can be achieved by 1) neglecting guidance dynamics, 2) restricting the motion in a plane, and 3) trajectory linearization.

It turns out that the attractive assumption, made by neglecting the dynamics of the pursuer, yields seriously misleading results.[1-3] As a consequence of this assumption, the direction of the optimal evasive maneuver is constant and is determined by the initial or terminal conditions. Moreover, if the pursuer's maneuverability is sufficient, the final miss distance is always zero.[3]

Most analytic studies in the past used two-dimensional models.[4-10] Whenever guidance dynamics were considered

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(even if by an approximation of a first-order time constant or a pure time delay) an oscillating or "bang-bang" structure of the optimal evasive maneuver became apparent. It was also shown that optimal evasion can guarantee nonzero miss distance even from a pursuer of unlimited maneuverability[7] or from one of an optimal guidance strategy.[9] It has been indicated, however, that the validity of 2-D analysis is limited to near "head-on" or "tail-chase" engagements.[10] For other initial conditions, 3-D analysis is required. The same study also demonstrated that, due to the "bang-bang" structure of the optimal evasive maneuvers, trajectory linearization is a good approximation for a wide range of parameters.

The objective of this paper, motivated by the above-mentioned results, is to analyze the problem of optimal missile avoidance using a three-dimensional linearized kinematic model. Analysis is based on the following set of assumptions: 1) Pursuer and evader are both considered as constant-speed mass points. 2) The pursuer is a homing missile launched against an initially non-maneuvering evader (target) in a collision course. 3) Relative pursuer-evader trajectory can be linearized around the intial line of sight. 4) Pursuer and evader both have perfect information on the relative state. Moreover, the evader has a complete knowledge of the pursuer's performance parameters. 5) Gravity can be neglected for both vehicles (not affecting relative trajectory). 6) The pursuing missile has two identical and independent guidance channels to execute proportional navigation in two perpendicular directions in a plane normal to the line of sight (true proportional navigation[11]). 7) The dynamics of each guidance channel is assumed to be (for sake of simplicity only) of first order. The validity of the first five assumptions and the effects of more complex pursuer dynamics are discussed in detail in Ref. 10.

Based on the above listed hypotheses, the 3-D missile avoidance is formulated as a fixed-duration optimal control problem maximizing a terminal payoff (the square of the miss distance). The control variable is the lateral-acceleration vector of the evading airplane. This acceleration is perpendicular to the velocity vector, its magnitude is bounded by the limit load factor (or maximum lift), and its direction is controlled by the airplane's roll-orientation.

First a mathematical model of unbounded missile maneuverability and infinite airplane roll-rate is used. This linear formulation leads to a closed-form solution and provides the basic insight into the problem. In consecutive steps, saturation of missile acceleration and realistic roll dynamics of the evading airplane are introduced.

The solutions obtained by the linearized 3-D model are compared both to the prediction of a 2-D linearized analysis[10] and to results of complete nonlinear (six degrees of freedom) simulation.

Mathematical Modeling

Three-Dimensional Vector Formulation

A three-dimensional pursuit-evasion is described by the vector equations

$$\dot{\vec{R}} = \vec{V}_T - \vec{V}_M \tag{1}$$

$$\vec{\Omega}_R = (\vec{R} \times \dot{\vec{R}}) / |\vec{R}|^2 \tag{2}$$

The acceleration command of the pursuing missile is given by assumption 6 as

$$(\dot{\vec{V}}_M)_c = K_N (\vec{\Omega}_R \times \vec{R}) / |\vec{R}| \tag{3}$$

while the actual acceleration is determined (see assumption 7 by

$$\tau \ddot{\vec{V}}_M + \dot{\vec{V}}_M = (\dot{\vec{V}}_M)_c \tag{4}$$

The acceleration of the constant-speed evader (the target) is normal to its vector velocity

$$\dot{\vec{V}}_T = (\vec{\Omega}_T \times \vec{V}_T) \tag{5}$$

Trajectory linearization around the initial collision course (ass. 2 & 3) yields

$$|\dot{\vec{R}}| \triangleq -V_R \simeq \text{const} \tag{6}$$

and as a consequence,

$$|\vec{R}(t)| = |\vec{R}_0| - V_R t = V_R (t_f - t) \tag{7}$$

determining the final time of the pursuit by

$$t_f = |\vec{R}_0| / V_R \tag{8}$$

Substituting Eqs. (6) and (7) into Eq. (3) and defining

$$K_N \triangleq N' V_R \tag{9}$$

yields

$$(\dot{\vec{V}}_M)_c = \frac{N'}{(t_f - t)^2} [\vec{R}(t) + (t_f - t)\dot{\vec{R}}(t)] \tag{10}$$

The system of differential equations (1), (4), (5), and the linearized feedback relation (10) determine the nine components of the vectors $\vec{R}(t)$, $\vec{V}_T(t)$, and $\vec{V}_M(t)$, if initial conditions and the target angular-velocity vector $\vec{\Omega}_T(t)$ are given. In the problem of optimal missile avoidance this last quantity is the control variable.

Nondimensional Scalar Equations—Linear Case

The initial collision plane (ass. 2) is taken as plane of reference for the direction of the vectors $\vec{R}$, $\vec{V}_M$ and $\vec{V}_T$ (see Fig. 1).

By choosing the X axis of the coordinate system to coincide with the initial line of sight, only those components of the relative motion which are normal to this direction have to be considered. The linearized equation of motion along this axis

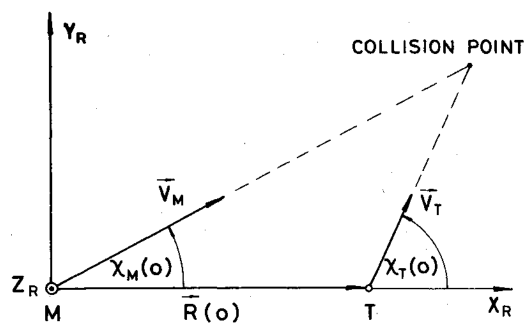

Fig. 1 Initial collision geometry.

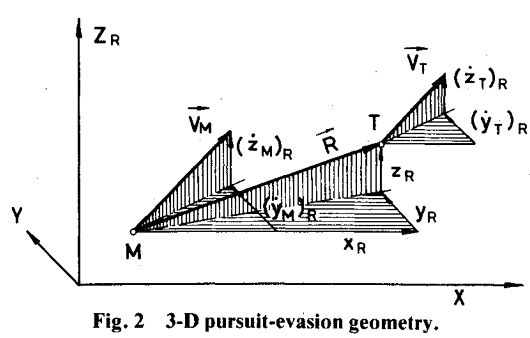

Fig. 2 3-D pursuit-evasion geometry.

is already solved by Eq. (7). The state vector is reduced to six components

$$x^T = (x_1 \ldots x_6) \triangleq (y, \dot{y}, \ddot{y}_M, z, \dot{z}, \ddot{z}_M) \tag{11}$$

y and z being the relative displacements perpendicular to the initial line of sight (see Fig. 2).

$$y = y_T - y_M$$
$$z = z_T - z_M \tag{12}$$

Moreover, if the Y and Z axes of the coordinate system are oriented to be in the missile maneuver planes (see Fig. 3), the state equation

$$\dot{x} = A(t)x + Bu \tag{13}$$

has a decoupled matrix $A(t)$

$$A(t) = \begin{bmatrix} A_1(t) & 0 \\ 0 & A_1(t) \end{bmatrix} \tag{14}$$

with

$$A_1(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \dfrac{N'}{\tau(t_f - t)^2} & \dfrac{N'}{\tau(t_f - t)} & \dfrac{-1}{\tau} \end{bmatrix} \tag{15}$$

The control vector u has two components,

$$u^T = (u_1, u_2) = (a_T \sin\phi_T, a_T \cos\phi_T) \tag{16}$$

a_T being the evading target lateral acceleration constrained by

$$0 \leq |a_T| \leq (a_T)_{\max} \tag{17}$$

The roll angle ϕ_T is measured relative to the collision plane. According to the convention used for airplanes, $\phi_T = 0$ means that the wings are parallel to the plane of reference (u_2 in Eq. (16) is in the Z_R direction). As a consequence, the transpose of the matrix B is given by

$$B^T = \begin{bmatrix} 0, & -\cos\chi_{T_0}\cos\phi_M, & 0, & 0, & \cos\chi_{T_0}\sin\phi_M, & 0 \\ 0, & \sin\phi_M, & 0, & 0, & \cos\phi_M, & 0 \end{bmatrix} \tag{18}$$

Use of nondimensional variables reduces the number of independent parameters and yields generalized results.[12] Introducing nondimensional time and distance by

$$\tilde{t} = t/\tau \tag{19}$$

$$\tilde{R} = R/\tau^2 (a_T)_{\max} \tag{20}$$

leads to normalization of velocity components by $\tau(a_T)_{\max}$ and acceleration by $(a_T)_{\max}$ As a result, the state equation

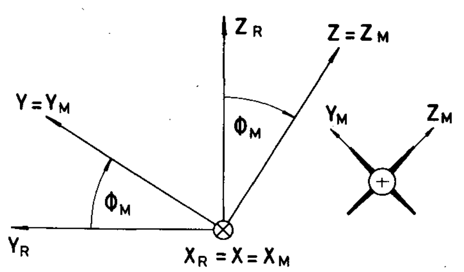

Fig. 3 Roll orientation of missile-maneuver planes.

(13) is transformed to

$$\mathrm{d}\tilde{x}/\mathrm{d}\tilde{t} = \tilde{A}(\tilde{t})\tilde{x} + B\tilde{u} \tag{21}$$

with a nondimensionalized state vector

$$\tilde{x}^T = \left(\tilde{y}, \frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{t}}, \frac{\mathrm{d}^2\tilde{y}_M}{\mathrm{d}\tilde{t}^2}, \tilde{z}, \frac{\mathrm{d}\tilde{z}}{\mathrm{d}\tilde{t}}, \frac{\mathrm{d}^2\tilde{z}_M}{\mathrm{d}\tilde{t}^2}\right) \tag{22}$$

and a normalized control vector

$$\tilde{u}^T = (\tilde{a}_T \sin\phi_T, \tilde{a}_T \cos\phi_T) \tag{23}$$

The decoupled structure of the state matrix is preserved with

$$\tilde{A}_1(\tilde{t}) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ \dfrac{N'}{(\tilde{t}_f - \tilde{t})^2} & \dfrac{N'}{(\tilde{t}_f - \tilde{t})} & -1 \end{bmatrix} \tag{24}$$

Nonlinear Effects

The state equation (13) or (21), describing system dynamics, is linear due to the implicit assumptions of unlimited missile maneuverability and infinite target roll-rate. A more realistic mathematical model has to consider the constraints on these variables. The state equation including such effects will no longer be linear.

Limited Missile Maneuverability

When missile-p aneuverability constraints are taken into account, it is necessary to redefine the state vector and the state equation. The components of the lateral acceleration, $\ddot{y}_M$ and $\ddot{z}_M$, are to be replaced by their required value $(\ddot{y}_M)_r$ and $(\ddot{z}_M)_r$ in nondimensional form

$$\tilde{x}_3 = (\ddot{y}_M)_r / (a_T)_{\max} \tag{25}$$

$$\tilde{x}_6 = (\ddot{z}_M)_r / (a_T)_{\max} \tag{26}$$

As these variables are not affected by the constraint, the differential equations for $\mathrm{d}\tilde{x}_3/\mathrm{d}\tilde{t}$ and $\mathrm{d}\tilde{x}_6/\mathrm{d}\tilde{t}$ remain unchanged.

The relation between the components of the acceleration, which are subject to constraints and their required values, can be expressed by the nonlinear saturation function defined as

$$\mathrm{sat}\left\{\frac{a}{b}\right\} \triangleq \begin{cases} a/b & \text{if } |a| < |b| \\ \mathrm{sign}[a/b] & \text{if } |a| \geq |b| \end{cases} \tag{27}$$

Consequently, the state equation will not have the linear form of Eq. (21) but has to be written as

$$\mathrm{d}\tilde{x}/\mathrm{d}\tilde{t} = F(\tilde{x}, \tilde{t}) + B\tilde{u} \tag{28}$$

For such saturation, two alternative formulations, expressed by two different vectograms, exist:

1) Circular (isotropic) vectogram (see Fig. 4) showing that the constraint of maneuverability applies to the resultant lateral acceleration

$$a_M^2 = \ddot{y}_M^2 + \ddot{z}_M^2 = (a_M)_{\max}^2 \,\mathrm{sat}\left\{\frac{(\ddot{y}_M)_r^2 + (\ddot{z}_M)_r^2}{(a_M)_{\max}^2}\right\} \tag{29}$$

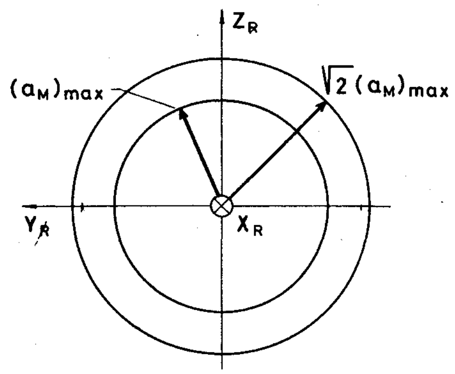

Fig. 4 Circular vectogram of missile acceleration.

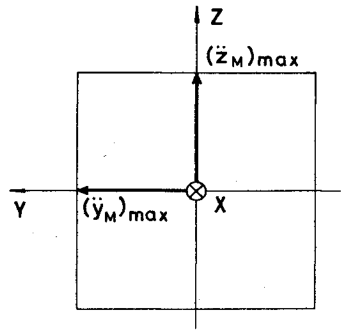

Fig. 5 Vectogram of cruiciform missile.

Such a vectogram represents the maneuverability of a thrust vector controlled (TVC) missile or of a cruciform configuration with unknown roll orientation. In this model it is assumed that saturation of both guidance channels takes place simultaneously.

By defining the missile-target maneuver ratio, which is one of the similarity parameters of the problem,[12] as

$$\mu \triangleq (a_M)_{max}/(a_T)_{max} \tag{30}$$

Eq. (29) can be written, using Eqs. (25) and (26), in a nondimensional form

$$\frac{\ddot{y}_M^2+\ddot{z}_M^2}{(a_T)^2_{max}} = \frac{d^2\tilde{y}_M}{d\tilde{t}^2}+\frac{d^2\tilde{z}_M}{d\tilde{t}^2} = \mu^2 \ \mathrm{sat}\left\{\frac{\tilde{x}_3^2+\tilde{x}_6^2}{\mu^2}\right\} \tag{31}$$

2) *Rectangular (square) vectogram* (see Fig. 5) indicating that saturation may occur in each guidance channel separately:

$$\ddot{y}_M = (\ddot{y}_M)_{max} \ \mathrm{sat}\left\{\frac{(\ddot{y}_M)_r}{(\ddot{y}_M)_{max}}\right\} \tag{32}$$

with a similar relation for $\ddot{z}_M$. A vectogram of this type represents a roll-stabilized cruciform missile with known roll orientation. Many surface-to-air missiles, equipped with linearly polarized antennas, have such characteristics. The perfect information set (see ass. 4) includes, when this type of missile is concerned, the direction of the roll orientation.

Defining the relative maneuverability of the guidance channels as

$$\mu_I \triangleq \frac{(\ddot{y}_M)_{max}}{(a_T)_{max}} = \frac{(\ddot{z}_M)_{max}}{(a_T)_{max}} \tag{33}$$

enables us to write Eq. (32) in nondimensional form

$$d^2\tilde{y}_M/d\tilde{t}^2 = \mu_I \ \mathrm{sat}\{\tilde{x}_3/\mu_I\} \tag{34}$$

A similar expression holds for the second channel.

For a rectangular vectogram, the actual missile-target maneuver ratio μ depends on the orientation of the target acceleration. It can be easily seen that

$$\mu_I \le \mu \le \sqrt{2}\mu_I \tag{35}$$

Limited Target Roll-Rate

A realistic description of the evader's roll dynamics requires that its roll orientation ϕ_T be considered not as a control but an additional state variable.

$$x_7 \triangleq \phi_T \tag{36}$$

The roll dynamics can be expressed by several alternative formulations:

1) The control variable is the target's roll-rate

$$\dot{x}_7 = P_T(\dot{\phi}_T)_{max} \tag{37}$$

subject to the constraint

$$|P_T| \le 1 \tag{38}$$

2) Closed-loop roll control based on the required roll orientation

$$(\dot{x}_7)_r = k_\phi[(x_7)_r - x_7] \tag{39}$$

subject to saturation

$$\dot{x}_7 = (\dot{\phi}_T)_{max} \ \mathrm{sat}\left\{\frac{\dot{x}_7}{(\dot{\phi}_T)_{max}}\right\} \tag{40}$$

3) Complete description of the evading airplane's roll dynamics controlled by an aerodynamic rolling moment

$$\ddot{x}_7 + \tau_\phi \dot{x}_7 = l_T \tag{41}$$

assuming both

$$|l_T| \le (l_T)_{max} \tag{42}$$

and

$$|\dot{x}_7| \le (\dot{\phi}_T)_{max} \tag{43}$$

The common feature in all these formulations is the limited roll-rate expressed by $(\dot{\phi}_T)_{max}$. In nondimensional form this constraint is expressed by

$$|d\tilde{x}_7/d\tilde{t}| \le \tau(\dot{\phi}_T)_{max} \triangleq \tilde{\phi}_T \tag{44}$$

defining $\tilde{\phi}_T$ as another similarity parameter of the problem.

Considering the target roll orientation ϕ_T as a state variable makes the overall dynamic system nonlinear even in the absence of missile saturation. In this case the state equation has the coupled form of

$$d\tilde{x}'/d\tilde{t} = f(\tilde{x}', \tilde{u}', \tilde{t}) \tag{45}$$

with a control vector $\tilde{u}'$ defined according to one of the alternative formulations.

Formulation of the Optimal Control Problem

The objective of missile avoidance is to maximize the survivability of the evading aircraft. Assuming a uniformly performing warhead and proximity fuse leads us to determine the payoff as the absolute value or the square of the miss distance. For linearized kinematics the last one is expressed as

$$m^2 = y^2(t_f) + z^2(t_f) \tag{46}$$

with t_f given by Eq. (8).

The optimal missile avoidance with a linearized kinematic model can therefore be formulated as a fixed-duration optimal control problem maximizing a terminal payoff (problem of Mayer).

Using nondimensional variables, the formulation (for unlimited missile maneuverability and target roll-rate) is the following:

Given the dynamic system described by Eq. (21) with zero initial conditions ($\tilde{x}_0 = 0$) and unspecified terminal state, find the optimal control $\tilde{u}^*(t)$ of the form of Eq. (23), subject to the constraint

$$0 \le \tilde{a}_T \le 1 \tag{47}$$

which maximizes the payoff

$$\tilde{J} = \tilde{x}_1^2(\tilde{t}_f) + \tilde{x}_4^2(\tilde{t}_f) \tag{48}$$

for the fixed value of $\tilde{t}_f$ given by

$$\tilde{t}_f = t_f/\tau = R_0/\tau V_R \tag{49}$$

If missile saturation and/or the target's roll dynamics are considered, a similar formulation holds with Eq. (21) replaced by Eq. (28) or Eq. (45), and stating the appropriate control structure with the additional constraints.

Formal Solutions

Linear Case

For the assumptions of unlimited missile maneuverability and infinite target roll-rate, stated in the previous section, the variational Hamiltonian

$$H(\tilde{x},\lambda,\tilde{u},\tilde{t}) = \lambda^T[\tilde{A}(\tilde{t})\tilde{x} + B\tilde{u}] \tag{50}$$

can be written, separating the part independent of the control variables, as

$$H = H_0(\tilde{x},\lambda,\tilde{t}) + \tilde{a}_T[\delta_1 \sin\phi_T + \delta_2 \cos\phi_T) \tag{51}$$

with

$$\delta_1 = \cos\chi_{T_0}(\lambda_5 \sin\phi_M - \lambda_2 \cos\phi_M) \tag{52}$$

$$\delta_2 = (\lambda_2 \sin\phi_M + \lambda_5 \cos\phi_M) \tag{53}$$

The optimal-control variables $\tilde{a}_T^*$ and ϕ_T^* have to maximize the Hamiltonian, yielding

$$(\tilde{a}_T)^* = \tfrac{1}{2}(\operatorname{sign} S_1 + 1) \tag{54}$$

with

$$S_1 \triangleq = \delta_1 \sin\phi_T + \delta_2 \cos\phi_T \tag{55}$$

and

$$(\phi_T)^* = \operatorname{tg}^{-1}[\delta_1/\delta_2] \tag{56}$$

Substituting Eq. (56) into Eq. (55) leads to

$$S_1 = k(\delta_1^2 + \delta_2^2) \ge 0 \tag{57}$$

k being a positive constant of proportionality determined by

$$k^2 = (\delta_1^2 + \delta_2^2)^{-1} \tag{58}$$

Equations (54) and (57) indicate directly that for optimal missile avoidance, maximum lateral load factor must always be used.

The components of the costate vector λ involved in S_1, δ_1, and δ_2 are determined by the adjoint equation

$$\mathrm{d}\lambda/\mathrm{d}\tilde{t} = -\partial H/\partial \tilde{x} \tag{59}$$

with the terminal conditions

$$\lambda(\tilde{t}_f) = -\partial \tilde{J}/\partial \tilde{x}|_{\tilde{t}_f} \tag{60}$$

resulting in

$$\lambda_1(\tilde{t}_f) = -2\tilde{y}(t_f) \triangleq -2\tilde{y}_f$$
$$\lambda_4(\tilde{t}_f) = -2\tilde{z}(t_f) \triangleq -2\tilde{z}_f$$
$$\lambda_i(\tilde{t}_f) = 0 \qquad (i=2,3,5,6) \tag{61}$$

Equation (59) yields for a linear system as Eq. (21)

$$\mathrm{d}\lambda/\mathrm{d}\tilde{t} = -\tilde{A}^T(\tilde{t})\lambda \tag{62}$$

Introducing the normalized time-to-go

$$\theta \triangleq \tilde{t}_f - \tilde{t} \tag{63}$$

leads to the transformation of Eqs. (60) and (62) to

$$\mathrm{d}\lambda/\mathrm{d}\theta = \tilde{A}^T(\theta)\lambda \tag{64}$$

with the initial conditions

$$\lambda_0 = -\partial \tilde{J}/\partial \tilde{x}|_{\theta=0} \tag{65}$$

The system of equations (64) can be reduced to two identical scalar differential equations of the form

$$\theta\left(\frac{\mathrm{d}^3\lambda_i}{\mathrm{d}\theta^3} + \frac{\mathrm{d}^2\lambda_i}{\mathrm{d}\theta^2}\right) + N'\frac{\mathrm{d}\lambda_i}{\mathrm{d}\theta} = 0 \quad (i=3,6) \tag{66}$$

which were solved in a closed form in Ref. 10. This solution combined with Eq. (65) yields

$$\lambda_i(\theta) = \lambda_{i_0} e^{-\theta}[1 - P_i(\theta)]$$
$$\lambda_{i+1}(\theta) = \lambda_{i_0}\theta e^{-\theta}[1 - P_{i+1}(\theta)]$$
$$\lambda_{i+2}(\theta) = \lambda_{i_0}\theta^2 e^{-\theta}[1 - P_{i+2}(\theta)] \qquad (i=1,4) \tag{67}$$

$P_i(\theta)$ are functions of θ depending on N' only. For integer values of N' these functions are polynomials of the order $(N'-2)$ (see Table A-1 in Ref. 10).

As a consequence of Eqs. (67) and (61), the time-dependent parts of λ_2 and λ_5 are identical:

$$\lambda_2(\tilde{t}) = -2\tilde{y}_f S_2(\tilde{t}_f - \tilde{t})$$
$$\lambda_5(\tilde{t}) = -2\tilde{z}_f S_2(\tilde{t}_f - \tilde{t}) \tag{68}$$

Substituting Eq. (68) into Eqs. (52) and (53) yields

$$\delta_1 = 2\cos\chi_{T_0}[\tilde{y}_f \cos\phi_M - \tilde{z}_f \sin\phi_M] S_2(\tilde{t}_f - \tilde{t}) \tag{69}$$

$$\delta_2 = -2[\tilde{y}_f \sin\phi_M + \tilde{z}_f \cos\phi_M] S_2(\tilde{t}_f - \tilde{t}) \tag{70}$$

The expressions in the brackets are the components of the miss-distance vector: the first in the plane of the initial collision and the second in the direction perpendicular to this plane.

As shown by Eqs. (54) and (57)

$$(\tilde{a}_T)^* = 1 \tag{71}$$

maximizing the Hamiltonian is equivalent to maximizing S_1. Inspection of Eqs. (57), (69), and (70) makes it obvious that for any given miss distance, maximization is achieved by

$$(\delta_1)^* = 0 \tag{72}$$

and as a consequence of Eq. (56),

$$\mathrm{tg}(\phi_T)^* = 0 \tag{73}$$

This indicates that the direction of the optimal evasive maneuver *has to be perpendicular to the plane of initial collision.* The optimal roll orientation, either $\phi_T^* = 0$ or $\phi_T^* = \pi$ is determined by the sign of the switch function S_2 which is identical to the one obtained in 2-D analysis.[10]

$$\cos[\phi_T(\tilde{t})]^* = -\mathrm{sign}[S_2(\tilde{t}_f - \tilde{t})] \tag{74}$$

Case of Circular Missile Vectogram

If missile maneuver constraints are taken into consideration, the linear state equation (21) is replaced with one of the nonlinear forms of Eq. (28). Nevertheless, this nonlinearity does not alter the structure of the optimal solution. As the controls appear separately in Eq. (28), the Hamiltonian preserves its separated form of Eq. (51). The control-dependent part is not affected by the saturation-type nonlinearity and the optimal control functions are also given in this case by Eqs. (52) and (53). Moreover, as saturation takes place in both guidance channels simultaneously, the time-dependent parts of λ_2 and λ_5 remain identical (although they are different from the costate variables of the linear case). As a consequence, both Eqs. (71) and (74) hold, yielding the same type of *"bang-bang" maneuver perpendicular to the initial collision plane* as for unlimited missile maneuverability. The switch function governing this maneuver is, however, different from the one obtained in closed form for the linear problem.

When missile saturation takes place in the terminal phase of the pursuit (it was shown[13] that it always occurs in this phase), the state equation of the system, and as a consequence, the adjoint equation, both change. As both components of missile acceleration are constants in the saturated phase, the submatrix $\tilde{A}_1(\tilde{t})$ in Eq. (24) is modified. For $\tilde{t} \geq \tilde{t}_s$, its second line will contain only zeros. The adjoint equations are given in this case as a function of normalized time-to-go θ, as follows:

$$\mathrm{d}\lambda_i/\mathrm{d}\theta = (N'/\theta^2)\lambda_{i+2} \tag{75}$$

$$\mathrm{d}\lambda_{i+1}/\mathrm{d}\theta = \lambda_i + (N'/\theta)\lambda_{i+2} \tag{76}$$

$$\mathrm{d}\lambda_{i+2}/\mathrm{d}\theta = -\lambda_{i+2} \qquad (i=1,4) \tag{77}$$

with the initial conditions of Eqs. (61).

From Eqs. (77) and (61) it is obvious that for $\theta \leq \theta_s$, λ_3 and λ_6 are both zero. This fact confirms that in the saturated phase the required accelerations have no influence on the solution. Consequently,

$$\lambda_i(\theta \leq \theta_s) = (\lambda_i)_0 = \mathrm{const} \tag{78}$$

$$\lambda_{i+1}(\theta \leq \theta_s) = (\lambda_i)_0 \cdot \theta \qquad (i=1,4) \tag{79}$$

It is easy to see that, due to the monotonity of λ_2 and λ_5, no switch can occur when the missile is saturated. It is also important to note that, as $F(\tilde{x},\tilde{t})$ in Eq. (28) has no discontinuity when saturation occurs, the costate variables remain continuous[14] at $\theta = \theta_s$.

The time of saturation is one of the unknowns and has to be determined with the complete solution of the costate variables, by solving a two-point boundary-value problem. Fortunately, due to the "bang-bang" type solution a simple and efficient search technique developed in Ref. 10 can be used as an alternative. The results obtained are identical with those of a previous 2-D analysis.[10] They show that the optimal switch function and the resulting miss distance both depend on the missile-target maneuver ratio μ. In Fig. 6 the switch function $S_2(\theta)$ for $\mu = 3$ is compared to the one obtained by a closed-form solution for infinite missile maneuverability.[10] The net effect of bounded missile acceleration is a longer terminal evasive maneuver. The miss distances are monotonically increasing with $1/\mu$. The dependence of the normalized miss distance can be roughly approximated (see Fig. 7 of Ref. 10) by

$$\tilde{m}^*(N', \tilde{t}_f, \mu) \cong \tilde{m}^*(N', \tilde{t}_f, \infty) + b(N', \tilde{t}_f)/\mu^2 \tag{80}$$

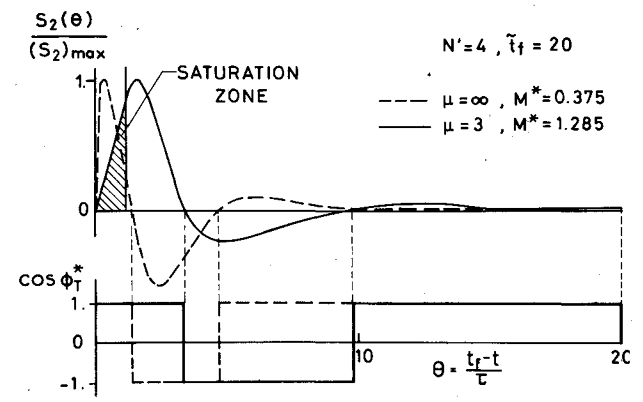

Fig. 6 Effect of limited missile maneuverability on the optimal switch function.

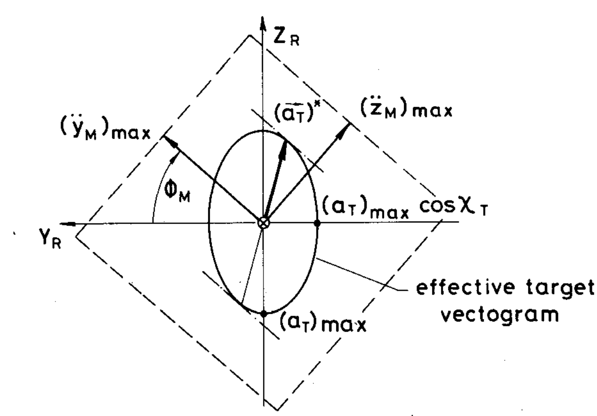

Fig. 7 Minimization of the effective missile-target maneuver ratio.

Case of Rectangular Missile Vectogram

In this case, representing a roll-stabilized cruciform missile with known roll orientation, saturation occurs in each guidance channel separately. The optimal control functions are determined, as for the previous case, by Eqs. (52) and (53). As Eq. (57) is not influenced by the saturation, Eq. (71) remains valid. However, the time-dependent parts of λ_2 and λ_5 are no longer identical and the optimal target roll orientation given by Eq. (56)

$$\mathrm{tg}(\phi_T)^* = \frac{\delta_1}{\delta_2} = \frac{\cos\chi_{T_0}[\lambda_5\sin\phi_M - \lambda_2\cos\phi_M]}{\lambda_2\sin\phi_M + \lambda_5\cos\phi_M} \tag{81}$$

cannot be determined by simple inspection.

It is, however, intuitively obvious, in view of Eq. (80), that for maximum miss distance, the effective missile-target maneuver ratio has to be minimum. This problem has a straightforward geometric solution shown in Fig. 7. It yields,

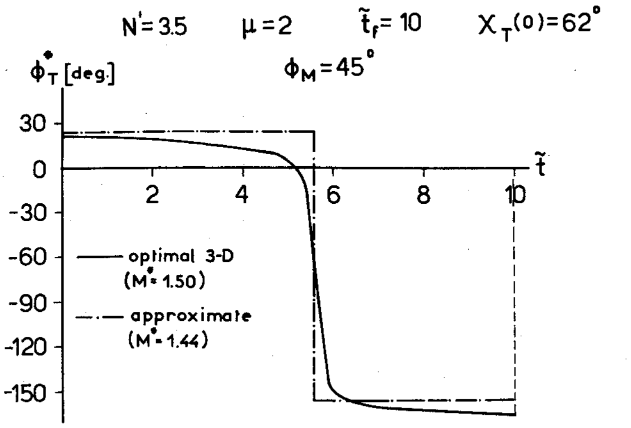

Fig. 8 Comparison of optimal and suboptimal avoidance maneuver from a roll-stabilized cruciform missile.

for $0<\phi_M<\pi/4$, which is the relevant one for cruciform missiles,

$$\mathrm{tg}(\phi_T^*)'=\cos\chi_{T_0}\ \mathrm{tg}\phi_M \tag{82}$$

This is a suboptimal solution, which maximizes the projection of target acceleration on the more susceptible guidance channel, but is can be easily implemented. Moreover, it takes definite advantage of the known missile roll orientation by always providing

$$\mu_{\mathrm{eff}}\leq\sqrt{2}\mu_I \tag{83}$$

The exact solution of Eq. (81), computed by a differential dynamic programming algorithm,[15] is depicted in Fig. 8. It shows almost constant values of ϕ_T^* [not very different from the prediction of Eq. (82)] for the initial and terminal phases connected by a very fast (but not instantaneous) transient roll maneuver. The miss distance achieved by this optimal maneuver is only slightly higher than the one obtained with the more simple intuitive suboptimal solution of Eq. (82).

Case of Limited Target Roll-Rate

The "bang-bang" solution obtained in the previously discussed cases assumes an infinite roll-rate of the evading aircraft. The solution of Eq. (81) also requires a very high roll-rate, which in most cases is beyond the actual performance limit of fighter aircraft. Whenever the real limitation on the target roll-rate is taken into account, the roll orientation ϕ_T becomes an additional state variable of the problem as indicated by Eq. (36). As a consequence, the system equation becomes nonlinear even in the absence of missile saturation. For such a case the optimal-control problem is formulated as follows:

Given the dynamic system with the state vector

$$\tilde{x}'=\mathrm{col}[\tilde{x}_1...\tilde{x}_7]=\mathrm{col}[\tilde{x},\phi_T] \tag{84}$$

The state equation has the form of Eq. (45). Its first six components are identical to Eq. (22) and the last equation is

$$\mathrm{d}\tilde{x}_7/\mathrm{d}\tilde{t}=\tilde{\phi}_T P_T \tag{85}$$

The system is controlled by

$$\tilde{u}'^T=[\tilde{a}_T,P_T] \tag{86}$$

The initial conditions $\tilde{x}_0'$ are given and the terminal state is not specified.

Find the optimal control $\tilde{\mathbf{u}}'^*(\tilde{t})$ subject to the constraints of Eqs. (47) and (38) that maximizes the terminal payoff of Eq. (48) for the fixed $\tilde{t}_f$ given by Eq. (49).

The variational Hamiltonian of the problem is

$$H'=\lambda^T f=H_0(\tilde{x},\lambda,\tilde{t})+\tilde{a}_T S_I(\tilde{x}_7,\lambda_2,\lambda_5)+P_T\tilde{\phi}_T\lambda_7 \tag{87}$$

with S_I written explicitly in Eq. (55).

The first optimal-control variable $\tilde{a}_T^*$ is given, as previously, by Eq. (71). The second control component maximizing the Hamiltonian has to be (if $\lambda_7\neq 0$)

$$(P_T)^*=\mathrm{sign}\lambda_7 \tag{88}$$

The time derivative of the new costate variable is

$$\mathrm{d}\lambda_7/\mathrm{d}\tilde{t}=-\partial H'/\partial\tilde{x}_7=-\tilde{a}_T\partial S_I/\partial\tilde{x}_7 \tag{89}$$

yielding (as a function of the normalized time-to-go)

$$\mathrm{d}\lambda_7/\mathrm{d}\theta=\tilde{a}_T[\delta_I(\lambda_2,\lambda_5)\cos\tilde{x}_7-\delta_2(\lambda_2,\lambda_5)\sin\tilde{x}_7] \tag{90}$$

with the initial condition $(\lambda_7)_0=0$.

A singular control is possible if $\lambda_7=\mathrm{d}\lambda_7/\mathrm{d}\theta=0$, requiring by Eq. (90)

$$\mathrm{tg}\tilde{x}_7=\mathrm{tg}\phi_T=\frac{\delta_I(\lambda_2,\lambda_5)}{\delta_2(\lambda_2,\lambda_5)} \tag{91}$$

Assuming that Eq. (91) holds, the singular value of P_T^* can be obtained from the second derivative. For the case of a circular vectogram, this value turns out to be zero. Comparing Eq. (91) with Eq. (56) indicates that the required roll orientation, predicted by Eq. (56) under the assumption of an infinite roll-rate, does not change by considering the roll-rate constraint.

Such steady-state ($P_T=0$, $\lambda_7=0$, $\mathrm{tg}\phi_T=\delta_I/\delta_2$) is, however, very unlikely. The mutual relations Eq. (85), (88), and (89) predict limit-cycle-type oscillations around the equilibrium value of Eq. (91). These oscillations damp out if higher-order roll dynamics are introduced in the model. Roll oscillations of small amplitude have no appreciable effect on the solution. The major effect is the reduction in the optimal miss distance as the value of the normalized maximum roll-rate $\tilde{\phi}_T$ decreases. This phenomenon was already predicted by the 2-D analysis (see Fig. 10 of Ref. 10).

Simulation Results

In order to validate the solution obtained with the 3-D linearized kinematic model used for the analysis, a comparison with results of a complete (nonlinear) simulation was carried out. The objectives of this effort were: 1) to determine the "domain of validity" of the linearized kinematic model, and 2) to justify the 3-D analysis.

A relatively extensive set of simulations has indicated that trajectory linearization, based on the initial collision course (see Fig. 1), yields valid results for optimal missile avoidance if two conditions are satisfied:

1) The dynamic similarity parameter[12] of the problem defined as

$$\tilde{\gamma}_T=(a_T)_{\max}\tau/V_T \tag{92}$$

is small (in the order of 0.15 rad or less)

2) The solution does not predict excessively long maneuvers in one direction. (As a rule of thumb the evader's direction change has to be limited to 30 deg.)

The first condition can be examined before the use of linearized kinematics is attempted, however the second one requires an a posteriori verification.

Justification of the 3-D analysis has been demonstrated by comparing the results of a 2-D analysis using the exact nonlinear kinematic model to those of the present 3-D

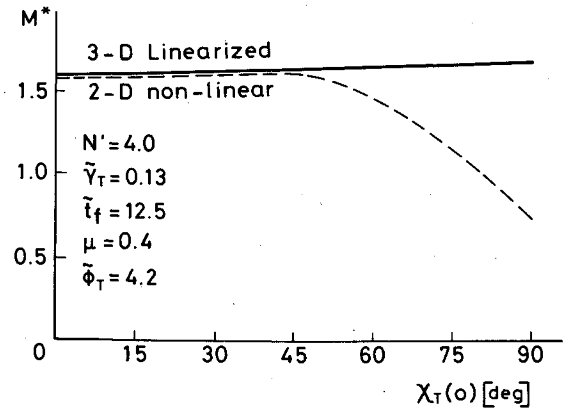

Fig. 9 Comparison of 3-D and 2-D tactics.

linearized study. The optimal control functions, obtained by both methods for a set of nondimensional parameters satisfying the conditions of linearity and for a wide range of initial angular conditions ($0<x_{T_0}<90$ deg), were used as inputs of the simulation program. The resulting normalized miss distances, plotted in Fig. 9, show that (at least for this example) the tactics suggested by the 3-D analysis have a definite advantage if the initial target azimuth angle x_{T_0} exceeds 45 deg.

Concluding Remarks

In this paper the problem of the 3-D optimal missile avoidance is analyzed in nondimensional form for realistic missile and aircraft models using linearized kinematics. The solution, derived by rigorous mathematical treatment, is presented in simple geometric terms, providing a clear insight into this inherently complex problem.

First, it is shown that the optimal evasion does not take place in the initial collision plane. Thus the effort in 3-D analysis is justified. Nevertheless, the optimal evasive maneuver is confined to a plane which, for a circular missile vectogram, is perpendicular to the initial plane of collision. In evading from a roll-stabilized cruciform missile, represented by a rectangular vectogram, further advantage can be taken by choosing a maneuver plane which minimizes the missile-target maneuver ratio.

The solution of the optimal control problem, maximizing the miss distance, is a "bang-bang" type maneuver with the continuous use of maximum load factor of the evading airplane. It can therefore be reduced to an optimal roll-position control problem of two consecutive phases: 1) orienting the airplane lateral-acceleration vector into the plane of optimal evasion, and 2) changing the direction of this acceleration, which has to be maximal, by rapid roll maneuvers of 180 deg in accordance with an optimal switch function.

The existence of an optimal maneuver plane enables us to use some results of the 2-D analysis[10] and as a consequence, avoids the solution of two-point boundary-value problems, which seems necessary a priori if missile saturation and limited airplane roll-rate are considered.

The optimal miss distances predicted by using a linearized kinematic model compare very well with results of a complete six-degree-of-freedom simulation.

Acknowledgments

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